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## Moment Determinacy of Powers and Products of Nonnegative Random Variables

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**Abstract:** We study the moment determinacy of nonlinear (Box-Cox) transformations of random variables. We establish new criteria for moment (in)determinacy and apply them to powers and products of nonnegative random variables. In particular, we show that the power and the product (of the same ‘order’) of generalized gamma random variables, both share the same moment determinacy property. Similar statement holds also for half-logistic random variables. The moment (in)determinacy of other transforms of random variables are also studied. We prove new results answering new questions in this area. In a few cases we either extend previously known results or provide new proofs to existing results. Related topics are briefly discussed.

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## 1. Introduction

The questions discussed in this paper can be described as follows. Suppose  $\xi$  is a nonnegative random variable defined on a given probability space  $(\Omega, \mathcal{F}, \mathbf{P})$  and  $\xi_1, \xi_2, \dots, \xi_n$  are its independent copies. Among the many transformations, of interest to us are the power, the product and the maximum, namely

$$X_n = \xi^n, \quad Y_n = \xi_1 \xi_2 \cdots \xi_n, \quad Z_n = \max\{\xi_1, \xi_2, \dots, \xi_n\},$$

as well as the equilibrium transformation.

We consider two possibilities: either  $n$  is a fixed positive integer, or  $n = N$  is a positive integer-valued random variable which is independent of all  $\xi$ 's.

Our goal is to study and characterize the moment (in)determinacy of the random variables  $X_n$ ,  $Y_n$  and  $Z_n$  on  $\mathbb{R}^+ = [0, \infty)$ , i.e., we deal with the Stieltjes moment problem. We establish relationships between the moment determinacy of these variables and find conditions under which they are uniquely determined by the moments, M-determinate, or that they are nonunique in terms of the moments, M-indeterminate. We use, respectively, the usual abbreviations M-det and M-indet. A few recent sources which the reader may need to consult regarding the moment problem for probability distributions are Berg [3], Lin and Stoyanov [15], Stoyanov and Lin [26] and Stoyanov [25].

To study random powers, products, etc., or more generally, Box-Cox transformations of random data, is a challenging probabilistic problem which is of independent interest. Results in this area are definitely important in contemporary stochastic modelling of real phenomena.

In this paper we deal with new problems and present new results with their proofs. We establish new and general criteria which are then applied to describe the moment (in)determinacy of specific random transformations. We also provide new proofs to some known results with reference to the original papers. Our results complement previous studies or represent different aspects of existing studies on this topic; see, e.g., DasGupta [6], Galambos and Simonelli [8], Berg [3], Ostrovska and Stoyanov [19] and Penson *et al.* [23].

The approach and the results in this paper can be further extended to distributions on the whole real line (Hamburger moment problem). Also, they can be used to characterize the moment determinacy properties of nonlinear transformations of some important sub-classes of distributions such as, e.g., the subexponential distributions; see Foss *et al.* [7].

The material is divided into relatively short sections each dealing with a specific question. All statements are followed by detailed proofs. The illustrations are based on commonly used distributions.

## 2. Moment determinacy of powers and products

We assume that  $n \geq 1$  is fixed. Since the ‘order’ of the power  $X_n = \xi^n$  and the ‘order’ of the product  $Y_n = \xi_1 \xi_2 \cdots \xi_n$  are the same, we might expect that these ‘similar’ random variables have similar properties. Let us ask two specific questions:

- (a) Do  $X_n$  and  $Y_n$  have the same number of finite moments?
- (b) Do  $X_n$  and  $Y_n$  share the same moment determinacy property?

In order to see that the answers are not straightforward, we start with a useful observation, Remark 1, and then provide one of our results, Proposition 1, below.

**Remark 1.** Note that for positive integer  $k$ ,  $\mathbf{E}[X_n^k] < \infty$  iff  $\mathbf{E}[\xi^{nk}] < \infty$ , but  $\mathbf{E}[Y_n^k] < \infty$  iff  $(\mathbf{E}[\xi^k])^n < \infty$  iff  $\mathbf{E}[\xi^k] < \infty$ . Therefore, for question (a),

$$\mathbf{E}[X_n^k] < \infty \implies \mathbf{E}[Y_n^k] < \infty$$

by Lyapunov's inequality. The converse, however, may not be true in general.

For example, consider the Pareto distribution  $F(x) = 1 - 1/x^2$ ,  $x \geq 1$ , and  $\xi \sim F$ . Then  $\mathbf{E}[Y_2] = (\mathbf{E}[\xi])^2 = 4 < \infty$ , but  $\mathbf{E}[X_2] = \mathbf{E}[\xi^2] = 2 \int_1^\infty (1/x) dx = \infty$ .

As for question (b), if  $\xi$  has a bounded support, then so does each of  $X_n$  and  $Y_n$ , and hence both  $X_n$  and  $Y_n$  have all moments finite and both are M-det.

These simple observations show that we have to study random variables with unbounded support contained in  $\mathbb{R}^+$ , and suppose that  $\xi$  has all moments finite:  $\mathbf{E}[\xi^k] \in (0, \infty)$  for all  $k = 1, 2, \dots$ . The latter implies that both  $X_n$  and  $Y_n$  have finite moments of all positive integer orders. Hence it is reasonable to ask whether  $X_n$  and  $Y_n$  are unique or nonunique in terms of their moments.

Let us mention first a special case. Suppose  $\xi$  obeys the standard exponential distribution. Then the power  $X_n = \xi^n$  is M-det iff the product  $Y_n = \xi_1 \xi_2 \cdots \xi_n$  is M-det iff  $n \leq 2$  (see, e.g., Berg [3] and Ostovska and Stoyanov [19]). This means that for any  $n = 1, 2, \dots$ , the power  $X_n$  and the product  $Y_n$  share the same moment determinacy property. Since Weibull distribution is just a power of the exponential distribution, it follows that if  $\xi$  obeys a Weibull distribution, then for any  $n = 1, 2, \dots$ , both  $X_n$  and  $Y_n$  also have the same moment determinacy property. Therefore, the answers to the above two questions (a) and (b) are positive for at least some special distributions including Weibull distributions. In this paper we will explore more distributions (see Theorem 6 and Section 11 below).

Note that in general, we have, by Lyapunov's inequality,

$$\mathbf{E}[X_n^s] = \mathbf{E}[\xi^{ns}] \geq (\mathbf{E}[\xi^s])^n = \mathbf{E}[Y_n^s] \quad \text{for all real } s > 0. \quad (1)$$

We use this moment inequality to establish a result which involves three of the most famous conditions for moment determinacy (Carleman's, Cramér's and Hardy's). For more details about Hardy's condition, see Stoyanov and Lin [26].

**Proposition 1.** (i) If the random variable  $\xi$  and the index  $n$  are such that  $X_n$  satisfies Carleman's condition (and hence is M-det), i.e.,  $\sum_{k=1}^\infty (\mathbf{E}[X_n^k])^{-1/(2k)} = \infty$ , then so does  $Y_n$ .

(ii) If  $X_n$  satisfies Cramér's condition (and hence is M-det), i.e.,  $\mathbf{E}[\exp(cX_n)] < \infty$  for some constant  $c > 0$ , then so does  $Y_n$ .

(iii) If  $X_n$  satisfies Hardy's condition (and hence is M-det), i.e.,  $\mathbf{E}[\exp(c\sqrt{X_n})] < \infty$  for some constant  $c > 0$ , then so does  $Y_n$ .

**Proof.** Part (i) follows immediately from (1). Parts (ii) and (iii) follow from the

fact that for each real  $s > 0$ ,

$$\mathbf{E}[\exp(cX_n^s)] = \sum_{k=0}^{\infty} \frac{c^k}{k!} \mathbf{E}[(X_n^s)^k] \geq \sum_{k=0}^{\infty} \frac{c^k}{k!} \mathbf{E}[(Y_n^s)^k] = \mathbf{E}[\exp(cY_n^s)].$$

The proof is complete.

**Corollary 1.** If  $\xi$  satisfies Cramér's condition, then both  $X_2$  and  $Y_2$  are M-det, and hence  $X_2$  and  $Y_2$  share the same moment determinacy property.

**Proof.** Note that  $\xi$  satisfies Cramér's condition iff  $X_2$  satisfies Hardy's condition. Then by Proposition 1(iii), both  $X_2$  and  $Y_2$  are M-det as claimed above.

### 3. Basic Example. Part (a)

Some of our results can be well illustrated by the generalized gamma distribution. We use the notation  $\xi \sim GG(\alpha, \beta, \gamma)$  if its density function is of the form

$$f(x) = cx^{\gamma-1}e^{-\alpha x^\beta}, \quad x \geq 0,$$

where  $\alpha, \beta, \gamma > 0$  and  $c$  is a norming constant,  $c = \beta\alpha^{\gamma/\beta}/\Gamma(\gamma/\beta)$ . It is known that the power  $X_n = \xi^n$  is M-det iff  $n \leq 2\beta$  (see, e.g., Pakes and Khattree [21]). We claim now that for  $n \leq 2\beta$ , the product  $Y_n = \xi_1\xi_2 \cdots \xi_n$  is also M-det. To see this, we note first that the density function  $h_n$  of the random variable  $\sqrt[n]{X_n}$  is

$$h_n(z) = \frac{2c}{n} z^{2\gamma/n-1} e^{-\alpha z^{2\beta/n}}, \quad z \geq 0.$$

This in turn implies that  $X_n$  satisfies Hardy's condition if  $2\beta/n \geq 1$ , so does  $Y_n$  for  $n \leq 2\beta$  by Proposition 1(iii).

To obtain further results, it is quite useful to write the explicit form of the density of the product  $Y_2 = \xi_1\xi_2$  when  $\xi$  has the generalized gamma distribution. This involves the function  $K_0(x)$ ,  $x > 0$ , the modified Bessel function of the second kind. Its definition and approximation are given as follows:

$$\begin{aligned} K_0(x) &= \frac{1}{2} \int_0^\infty t^{-1} e^{-t-x^2/(4t)} dt, \quad x > 0, \\ &= \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} \left[1 - \frac{1}{8x} \left(1 - \frac{9}{16x} \left(1 - \frac{25}{24x}\right)\right) + o(x^{-3})\right] \quad \text{as } x \rightarrow \infty \end{aligned}$$

(see, e.g., Glasser *et al.* [9] and Malham [16], pp. 37–38).

**Lemma 1.** (See also Malik [17].) Let  $Y_2 = \xi_1\xi_2$ , where  $\xi_1$  and  $\xi_2$  are independent random variables having the same distribution  $GG(\alpha, \beta, \gamma)$  with density  $f(x) = cx^{\gamma-1}e^{-\alpha x^\beta}$ ,  $x > 0$ , and  $c = \beta\alpha^{\gamma/\beta}/\Gamma(\gamma/\beta)$ . Then the density function  $g_2$  of  $Y_2$  is

$$\begin{aligned} g_2(x) &= \frac{2c^2}{\beta} x^{\gamma-1} K_0(2\alpha x^{\beta/2}), \quad x > 0, \\ &\approx Cx^{\gamma-\beta/4-1} e^{-2\alpha x^{\beta/2}}, \quad \text{as } x \rightarrow \infty. \end{aligned}$$

**Proof.** (Method I) Let  $G_2$  be the distribution function of  $Y_2$ . Then

$$\overline{G}_2(x) := 1 - G_2(x) = \mathbf{P}[Y_2 > x] = \int_0^\infty \mathbf{P}[\xi_1 > x/y] c y^{\gamma-1} e^{-\alpha y^\beta} dy,$$

and hence the density function of  $Y_2$  is

$$\begin{aligned} g_2(x) &= c^2 x^{\gamma-1} \int_0^\infty y^{-1} e^{-\alpha x^\beta / y^\beta - \alpha y^\beta} dy = \frac{c^2}{\beta} x^{\gamma-1} \int_0^\infty t^{-1} e^{-t - (\alpha^2 x^\beta)/t} dt \\ &= \frac{2c^2}{\beta} x^{\gamma-1} K_0(2\alpha x^{\beta/2}), \quad x > 0. \end{aligned}$$

(Method II) We can use the moment function (or Mellin transform) because it uniquely determines the corresponding distribution. To do this, we note that

$$\mathbf{E}[Y_2^s] = (\mathbf{E}[\xi_1^s])^2, \quad \mathbf{E}[\xi_1^s] = c \Gamma((\gamma + s)/\beta) (\beta \alpha^{(\gamma+s)/\beta})^{-1}, \quad \text{and}$$

$$\int_0^\infty x^s K_0(x) dx = 2^{s-1} (\Gamma((s+1)/2))^2 \quad \text{for all } s > 0$$

(see, e.g., Gradshteyn and Ryzhik [10], p. 676, formula 6.561(16)). Detailed calculation is omitted.

It may look surprising that several commonly used distributions are related to the Bessel function in such a natural way as in Lemma 1. Such an observation may open a new topic in distribution theory. For example, if  $\xi$  is a half-normal random variable, i.e.,  $\xi \sim GG(1/2, 2, 1)$  with the density  $f(x) = \sqrt{2/\pi} e^{-x^2/2}$ ,  $x \geq 0$ , then  $Y_2$  has the density function  $g_2(x) = (2/\pi) K_0(x) \approx C_2 x^{-1/2} e^{-x}$  as  $x \rightarrow \infty$ , with the moment function  $\mathcal{M}(s) = \mathbf{E}[Y_2^s] = (2^s/\pi) \Gamma^2((s+1)/2)$ ,  $s > -1$ . The distribution of  $Y_2 = \xi_1 \xi_2$  may be called the half-Bessel distribution and its symmetric counterpart with density  $h_2(x) = (1/\pi) K_0(x)$ ,  $x \in \mathbb{R} = (-\infty, \infty)$ , is called the standard Bessel distribution (note that  $K_0$  is an even function and  $h_2$  happens to be the density of the product of two independent standard normal random variables; see also DasGupta [6]). It can be checked that for real  $s > 0$  we have  $(\mathbf{E}[(Y_2^s)^n])^{-1/(2n)} \approx C_s n^{-s/2}$  as  $n \rightarrow \infty$ , and hence  $Y_2^s$  satisfies Carleman's condition iff  $s \leq 2$ . Actually, it follows from the density function  $g_2$  and its asymptotic behavior that  $Y_2$  satisfies Cramér's condition. Therefore, by Hardy's criterion, the square of  $Y_2$ , i.e.,  $Y_2^2 = \xi_1^2 \xi_2^2$ , is M-det.

Let us express the latter by words: The square of the product of two independent half-normal random variables is M-det. One additional conclusion is that, the product of two independent  $\chi^2$ -distributed random variables is M-det. In addition, these properties can be compared with the known fact that the power 4 of a normal random variable is M-det (see, e.g., Berg [1] or Stoyanov [24]).

#### 4. A slow growth rate of the moments implies moment determinacy

It is known and well understood that the moment determinacy of a distribution depends on the rate of growing of the moments. Let us establish first a result which

is of independent interest and we will show how to use it and make conclusions about powers and products of random variables.

Suppose  $X \sim F$  is a nonnegative random variable. Denote by  $m_k = \mathbf{E}[X^k]$  the moment of  $X$  of order  $k$ . Assume  $m_1 > 0$ , meaning that  $X$  is not a degenerate random variable at 0. Further, assume  $m_k < \infty$  for all  $k = 1, 2, \dots$ . We want to clarify the relationship between the growth rate of the moments and the moment determinacy of  $X$ , or of  $F$ .

**Lemma 2.** For each  $k \geq 1$ , we have

- (i)  $\log m_k \leq \log m_{k+1}$ , and
- (ii)  $m_1 m_k \leq m_{k+1}$ .

**Proof.** By Lyapunov's inequality, we have  $(m_k)^{1/k} \leq (m_{k+1})^{1/(k+1)}$ . Therefore,

$$\frac{1}{k} \log m_k \leq \frac{1}{k+1} \log m_{k+1} \leq \frac{1}{k} \log m_{k+1},$$

and hence  $\log m_k \leq \log m_{k+1}$ . This proves part (i).

To prove part (ii), let us consider  $m_1 \leq (m_k)^{1/k} \leq (m_{k+1})^{1/(k+1)}$ , and hence

$$m_1 m_k \leq (m_k)^{1/k} m_k = m_k^{(k+1)/k} \leq m_{k+1}.$$

This completes the proof.

Lemma 2 shows that  $\log m_k$  is increasing in  $k$  and that the ratio  $m_{k+1}/m_k$  has a lower bound  $m_1$  whatever the nonnegative random variable  $X$  is. The next theorem provides the upper bound of  $m_{k+1}/m_k$ , or, equivalently, the growth rate of  $\log m_k$ , for which  $X$  is M-det.

**Theorem 1.** Let  $m_{k+1}/m_k = O((k+1)^2)$  as  $k \rightarrow \infty$ , or, equivalently, for some real constant  $c^*$ ,  $\log m_{k+1} \leq \log m_k + 2 \log(k+1) + c^*$  for all large  $k$ . Then  $X$  satisfies Carleman's condition and is M-det.

**Proof.** By the assumption, there exists a constant  $C > 0$  such that

$$m_k^{(k+1)/k} \leq m_{k+1} \leq C(k+1)^2 m_k \text{ for all large } k,$$

which implies

$$m_k^{1/k} \leq C(k+1)^2 \text{ for all large } k,$$

and hence

$$m_k^{-1/(2k)} \geq C^{-1/2} (k+1)^{-1} \text{ for large } k.$$

Therefore,  $X$  satisfies Carleman's condition  $\sum_{k=1}^{\infty} m_k^{-1/(2k)} = \infty$ , and is M-det. The proof is complete.

We can slightly extend Theorem 1 as follows. For a real number  $a$  we denote by  $[a]$  the largest integer less than or equal to  $a$ .

**Theorem 1'.** Suppose there is a real number  $a \geq 1$  such that the moments of the random variable  $X$  satisfy the condition  $m_{k+1}/m_k = O((k+1)^{2/a})$  as  $k \rightarrow \infty$ .

Then the power  $X^{[a]}$  satisfies Carleman's condition and is M-det.

**Proof.** Note that

$$\begin{aligned}\frac{\mathbf{E}[(X^{[a]})^{k+1}]}{\mathbf{E}[(X^{[a]})^k]} &= \frac{\mathbf{E}[X^{[a]k+[a]}]}{\mathbf{E}[X^{[a]k+[a]-1}]} \frac{\mathbf{E}[X^{[a]k+[a]-1}]}{\mathbf{E}[X^{[a]k+[a]-2}]} \cdots \frac{\mathbf{E}[X^{[a]k+1}]}{\mathbf{E}[X^{[a]k}]} \\ &= O((k+1)^{(2/a)[a]}) = O((k+1)^2) \text{ as } k \rightarrow \infty.\end{aligned}$$

Hence, by Theorem 1,  $X^{[a]}$  satisfies Carleman's condition and is M-det.

**Theorem 2.** Let  $\xi, \xi_i, i = 1, 2, \dots, n$ , be defined as before and  $Y_n = \xi_1 \cdots \xi_n$ . If  $\xi$  and the index  $n$  are such that

$$\mathbf{E}[\xi^{k+1}]/\mathbf{E}[\xi^k] = O((k+1)^{2/n}) \text{ as } k \rightarrow \infty,$$

then  $Y_n$  satisfies Carleman's condition and is M-det.

**Proof.** By the assumption, we have

$$\mathbf{E}[Y_n^{k+1}]/\mathbf{E}[Y_n^k] = (\mathbf{E}[\xi^{k+1}]/\mathbf{E}[\xi^k])^n = O((k+1)^2) \text{ as } k \rightarrow \infty.$$

This, according to Theorem 1, implies the validity of Carleman's condition for  $Y_n$ , hence  $Y_n$  is M-det as stated above.

**Theorem 2'.** Let  $a \geq 1$ . If

$$\mathbf{E}[\xi^{k+1}]/\mathbf{E}[\xi^k] = O((k+1)^{2/a}) \text{ as } k \rightarrow \infty,$$

then  $Y_{[a]}$  satisfies Carleman's condition and is M-det.

**Proof.** Note that

$$\begin{aligned}\mathbf{E}[Y_{[a]}^{k+1}]/\mathbf{E}[Y_{[a]}^k] &= (\mathbf{E}[\xi^{k+1}]/\mathbf{E}[\xi^k])^{[a]} \\ &= O((k+1)^{(2/a)[a]}) = O((k+1)^2) \text{ as } k \rightarrow \infty.\end{aligned}$$

The conclusions follow from Theorem 1. The proof is complete.

## 5. Part (b) of Basic Example

We now give an alternative proof of the moment determinacy established in Part (a) of Basic Example; see Section 3.

Let  $\xi \sim GG(\alpha, \beta, \gamma)$  with the density  $f(x) = cx^{\gamma-1}e^{-\alpha x^\beta}$ ,  $x \geq 0$ , where  $\alpha, \beta, \gamma > 0$  and  $c = \beta\alpha^{\gamma/\beta}/\Gamma(\gamma/\beta)$ . We claim that for  $n \leq 2\beta$ , both  $X_n = \xi^n$  and  $Y_n = \xi_1\xi_2 \cdots \xi_n$  are M-det. To see this, we first calculate

$$\frac{\mathbf{E}[X_n^{k+1}]}{\mathbf{E}[X_n^k]} = \frac{\mathbf{E}[\xi^{n(k+1)}]}{\mathbf{E}[\xi^{nk}]} = \frac{\Gamma((\gamma + n(k+1))/\beta)}{\alpha^{n/\beta}\Gamma((\gamma + nk)/\beta)} \approx (n/\alpha\beta)^{n/\beta}(k+1)^{n/\beta} \text{ as } k \rightarrow \infty.$$

For this relation we use the approximation of the gamma function:

$$\Gamma(x) \approx \sqrt{2\pi}x^{x-1/2}e^{-x} \text{ as } x \rightarrow \infty$$



(see, e.g., Whittaker and Watson [27], p. 253). Then by Theorem 1,  $X_n$  is M-det if  $n \leq 2\beta$ , and by Theorem 2,  $Y_n$  is M-det if  $1/\beta \leq 2/n$  or if  $n \leq 2\beta$ , because  $\mathbf{E}[\xi^{k+1}]/\mathbf{E}[\xi^k] = O((k+1)^{1/\beta})$  as  $k \rightarrow \infty$ .

For example, if  $\xi \sim \text{Exp}(1) = GG(1, 1, 1)$ , the standard exponential distribution, then the product  $Y_2 = \xi_1 \xi_2$  is M-det. In fact, by Lemma 1, the density function  $g_2$  of  $Y_2$  is  $g_2(x) = 2K_0(2\sqrt{x}) \approx Cx^{-1/4}e^{-2\sqrt{x}}$  as  $x \rightarrow \infty$ , where  $K_0$  is the modified Bessel function of the second kind (see also Malik and Trudel [18], p. 417, and Gradshteyn and Ryzhik [10], p. 917, formula 8.432(8)). If  $\xi \sim GG(1/2, 2, 1)$ , the half-normal distribution, then  $Y_n = \xi_1 \xi_2 \cdots \xi_n$  is M-det for  $n \leq 4$ . As mentioned before, the density function of the product of two half-normals is  $g_2(x) = (2/\pi)K_0(x) \approx C_2x^{-1/2}e^{-x}$  as  $x \rightarrow \infty$ .

## 6. More results related to Theorems 1 and 2

Under the same assumption as that in Theorem 1, we even have a stronger statement; see Theorem 3 below. Note that its proof does not use Lyapunov's inequality, and that Hardy's condition implies Carleman's condition. For convenience, we recall in the next lemma a characterization of Hardy's condition in terms of the moments (see Stoyanov and Lin [26], Theorem 3).

**Lemma 3.** Let  $a \in (0, 1]$  and let  $X$  be a nonnegative random variable. Then  $\mathbf{E}[\exp(cX^a)] < \infty$  for some constant  $c > 0$  iff  $\mathbf{E}[X^k] \leq c_0^k \Gamma(k/a + 1)$ ,  $k = 1, 2, \dots$ , for some constant  $c_0 > 0$  (independent of  $k$ ). In particular,  $X$  satisfies Hardy's condition, i.e.,  $\mathbf{E}[\exp(c\sqrt{X})] < \infty$  for some constant  $c > 0$ , iff  $\mathbf{E}[X^k] \leq c_0^k (2k)!$ ,  $k = 1, 2, \dots$ , for some constant  $c_0 > 0$  (independent of  $k$ ).

**Theorem 3.** Suppose  $X$  is a nonnegative random variable with finite moments  $m_k = \mathbf{E}[X^k]$ ,  $k = 1, 2, \dots$ , such that the condition in Theorem 1 holds:  $m_{k+1}/m_k = O((k+1)^2)$  as  $k \rightarrow \infty$ . Then  $X$  satisfies Hardy's condition, and is M-det.

**Proof.** By the assumption, there exists a constant  $c_* \geq m_1 > 0$  such that

$$m_{k+1} \leq c_*(k+1)^2 m_k \quad \text{for } k = 0, 1, 2, \dots,$$

where  $m_0 \equiv 1$ . This implies that

$$m_{k+1} \leq (c_*/2)(2k+2)(2k+1)m_k \quad \text{for } k = 0, 1, 2, \dots,$$

and hence  $m_{k+1} \leq (c_*/2)^{k+1} \Gamma(2k+3) m_0$  for  $k = 0, 1, 2, \dots$ . Taking  $c_0 = c_*/2$ ,

$$m_{k+1} \leq c_0^{k+1} \Gamma(2k+3) \quad \text{for } k = 0, 1, 2, \dots,$$

or, equivalently,

$$m_k \leq c_0^k \Gamma(2k+1) \quad \text{for } k = 1, 2, \dots$$

Hence  $X$  satisfies Hardy's condition by Lemma 3. This completes the proof.

**Remark 2.** The constant 2 (the growth rate of the moments) in the condition of Theorem 1 is the best possible. This means that for each  $\varepsilon > 0$ , there exists a random variable  $X$  such that  $m_{k+1}/m_k = O((k+1)^{2+\varepsilon})$  as  $k \rightarrow \infty$ , but  $X$  is

M-indet. To see this, let us consider  $X = \xi \sim GG(1, \beta, 1)$ , which has density  $f(x) = c \exp(-x^\beta)$ ,  $x > 0$ . We have

$$\frac{\mathbf{E}[\xi^{k+1}]}{\mathbf{E}[\xi^k]} = \frac{\Gamma((k+2)/\beta)}{\Gamma((k+1)/\beta)} \approx \beta^{-1/\beta} (k+1)^{1/\beta} \text{ as } k \rightarrow \infty.$$

If for  $\varepsilon > 0$  we take  $\beta = \frac{1}{2+\varepsilon} < \frac{1}{2}$ , then  $\mathbf{E}[\xi^{k+1}]/\mathbf{E}[\xi^k] = O((k+1)^{2+\varepsilon})$  as  $k \rightarrow \infty$ . However, as mentioned before,  $X$  is M-indet.

**Remark 3.** The constant  $2/n$  in the condition of Theorem 2 is the best possible. Indeed, we can show that for each  $\varepsilon > 0$ , there exists a random variable  $\xi$  such that  $\mathbf{E}[\xi^{k+1}]/\mathbf{E}[\xi^k] = O((k+1)^{2/n+\varepsilon})$  as  $k \rightarrow \infty$ , but  $Y_n$  is M-indet. To see this, let us consider  $X = \xi \sim GG(1, \beta, 1)$ . For each  $\varepsilon > 0$ , take  $\beta = 1/(2/n + \varepsilon)$ , then

$$\frac{\mathbf{E}[\xi^{k+1}]}{\mathbf{E}[\xi^k]} = \frac{\Gamma((k+2)/\beta)}{\Gamma((k+1)/\beta)} = O((k+1)^{2/n+\varepsilon}) \text{ as } k \rightarrow \infty.$$

However, since  $n > 2\beta$ ,  $Y_n$  is M-indet (see Part (d) of Basic Example, Section 10).

## 7. Faster growth rate of the moments implies moment indeterminacy

We now establish a result which is converse to Theorem 1.

**Theorem 4.** Suppose  $X$  is a nonnegative random variables with finite moments  $m_k = \mathbf{E}[X^k]$ ,  $k = 1, 2, \dots$ , such that  $m_{k+1}/m_k \geq C(k+1)^{2+\varepsilon}$  for all large  $k$ , where  $C$  and  $\varepsilon$  are positive constants. Assume further that  $X$  has a density function  $f$  satisfying the condition: for some  $x_0 > 0$ ,  $f$  is positive and differentiable on  $[x_0, \infty)$  and

$$L_f(x) := -\frac{xf'(x)}{f(x)} \nearrow \infty \text{ as } x_0 < x \rightarrow \infty. \quad (2)$$

Then  $X$  is M-indet.

**Proof.** Without loss of generality we can assume that  $m_{k+1}/m_k \geq C(k+1)^{2+\varepsilon}$  for each  $k \geq 1$ . Therefore,

$$m_{k+1} \geq C^k ((k+1)!)^{2+\varepsilon} m_1 \text{ for } k = 1, 2, \dots$$

Taking  $C_0 = \min\{C, m_1\}$ , we have

$$m_{k+1} \geq C_0^{k+1} ((k+1)!)^{2+\varepsilon} \text{ for } k = 1, 2, \dots,$$

or, equivalently,

$$m_k \geq C_0^k (k!)^{2+\varepsilon} = C_0^k (\Gamma(k+1))^{2+\varepsilon} \text{ for } k = 2, 3, \dots$$

Since  $\Gamma(x+1) = x\Gamma(x) \approx \sqrt{2\pi} x^{x+1/2} e^{-x}$  as  $x \rightarrow \infty$ , we have that for some constant  $c > 0$ ,

$$m_k^{-1/(2k)} \leq C_0^{-1/2} (\Gamma(k+1))^{-(2+\varepsilon)/(2k)} \approx ck^{-1-\varepsilon/2} \text{ for all large } k.$$

This implies that the Carleman quantity of  $f$  is finite:

$$\mathbf{C}[f] := \sum_{k=1}^{\infty} m_k^{-1/(2k)} < \infty.$$

We sketch the rest of the proof. Following the proof of Theorem 3 in Lin [12], we first construct a symmetric distribution  $G$  on  $\mathbb{R}$ , obeyed by a random variable  $Y$ , such that  $\mathbf{E}[Y^{2k}] = \mathbf{E}[X^k]$ ,  $\mathbf{E}[Y^{2k-1}] = 0$  for  $k = 1, 2, \dots$ . Let  $g$  be the density of  $G$ . Then

$$\mathbf{C}[g] := \sum_{k=1}^{\infty} (\mathbf{E}[Y^{2k}])^{-1/(2k)} = \sum_{k=1}^{\infty} (\mathbf{E}[X^k])^{-1/(2k)} = \mathbf{C}[f] < \infty.$$

This implies that for some  $x_0^* > x_0$ , the logarithmic integral (Krein quantity of  $g$ ) over  $\{x : |x| \geq x_0^*\}$  is finite:

$$\mathbf{K}[g] := \int_{|x| \geq x_0^*} \frac{-\log g(x)}{1+x^2} dx < \infty,$$

as shown in the proof of Theorem 2 in Lin [12]. Finally, according to a result of Pedersen [22], this is a sufficient condition for  $Y$  to be M-indet on  $\mathbb{R}$ , and hence  $X$  is M-indet on  $\mathbb{R}^+$  (for details, see Pakes [20], Proposition 1 and Theorem 3). The proof is complete.

## 8. Part (c) of Basic Example

Let  $\xi \sim GG(\alpha, \beta, \gamma)$  with the density  $f(x) = cx^{\gamma-1}e^{-\alpha x^\beta}$ ,  $x \geq 0$ , where  $\alpha, \beta, \gamma > 0$  and  $c = \beta\alpha^{\gamma/\beta}/\Gamma(\gamma/\beta)$ . Then for  $n > 2\beta$ ,  $X_n = \xi^n$  is M-indet. To see this, recall that

$$\frac{\mathbf{E}[X_n^{k+1}]}{\mathbf{E}[X_n^k]} \approx (n/\alpha\beta)^{n/\beta} (k+1)^{n/\beta} \quad \text{as } k \rightarrow \infty,$$

where  $n/\beta > 2$ . The density function  $h$  of  $X_n$  satisfies the condition (2):

$$L_h(x) := -\frac{xh'(x)}{h(x)} = -\frac{\gamma}{n^2} + \frac{\alpha\beta}{n^2}x^{\beta/n} \nearrow \infty \quad \text{ultimately as } x \rightarrow \infty.$$

Therefore, for  $n > 2\beta$ ,  $X_n$  is M-indet by Theorem 4.

**Remark 4.** To use Theorem 4 is another way to prove some known facts, for example, that the log-normal distribution and the cube of the exponential distribution are M-indet. Indeed, for  $X \sim \text{LogN}(0, 1)$ , we have the moment recurrence

$$m_{k+1} = e^{k+1/2}m_k, \quad k = 1, 2, \dots,$$

and for  $X = \xi^3$ , where  $\xi \sim \text{Exp}(1)$ , we have

$$m_{k+1} = (3k+1)(3k+2)(3k+3)m_k, \quad k = 1, 2, \dots$$

It is easily seen that in both cases the growth rates of the moments are quite fast. For the cube of  $Exp(1)$  we have  $m_{k+1}/m_k \geq C(k+1)^3$ ,  $k = 1, 2, \dots$ , for some constant  $C > 0$ , so the rate is more than 2. For  $LogN$  the rate is exponential, hence much larger than 2. It remains to check that condition (2) is satisfied for the density of  $\xi^3$  and the density of  $LogN$ . Details are omitted.

We can make one step more by considering the logarithmic skew-normal distributions with density  $f_\lambda(x) = (2/x)\varphi(\ln x)\Phi(\lambda \ln x)$ ,  $x > 0$ , where  $\lambda$  is a real number. (When  $\lambda = 0$ ,  $f_\lambda$  reduces to the standard log-normal density.) Then we have the moment relationship

$$m_{k+1} \approx e^{(k+1/2)\rho} m_k, \quad \text{as } k \rightarrow \infty,$$

where  $\rho \in (0, 1]$  is a constant (see, e.g., Lin and Stoyanov [15], Proposition 3). Thus the moments grow very fast, exponentially, and it remains to check that the density function  $f_\lambda$  satisfies the condition (2):

$$L_{f_\lambda}(x) := -\frac{x f'_\lambda(x)}{f_\lambda(x)} \nearrow \infty \quad \text{ultimately as } x \rightarrow \infty.$$

Therefore, by the above Theorem 4, we conclude that all logarithmic skew-normal distributions are M-indet. This is one of the results in Lin and Stoyanov [15] where a different proof is given.

## 9. The M-indet property of the product $Y_n = \xi_1 \xi_2 \cdots \xi_n$

In the next theorem we describe conditions on the distribution of  $\xi$  under which the product  $Y_n = \xi_1 \xi_2 \cdots \xi_n$  is M-indet.

**Theorem 5.** Let  $\xi \sim F$ , where  $F$  is absolutely continuous with density  $f > 0$  on  $\mathbb{R}^+$  and has finite moments of all positive integer orders. Assume further that:

- (i)  $f$  is decreasing in  $x \geq 0$ , and
- (ii) there exist two constants  $x_0 \geq 1$  and  $A > 0$  such that

$$f(x)/\overline{F}(x) \geq A/x \quad \text{for } x \geq x_0, \tag{3}$$

and some constants  $B > 0$ ,  $\alpha > 0$ ,  $\beta > 0$  and a real  $\gamma$  such that

$$\overline{F}(x) \geq Bx^\gamma e^{-\alpha x^\beta} \quad \text{for } x \geq x_0. \tag{4}$$

Then, for  $n > 2\beta$ , the product  $Y_n$  has a finite Krein quantity and is M-indet.

**Corollary 2.** Let  $\xi \sim F$  satisfy the conditions in Theorem 5 with  $\beta < 1/2$ . Then  $F$  itself is M-indet.

**Lemma 4.** Under the condition (3), we have

$$\int_x^\infty \frac{f(u)}{u} du \geq \frac{A}{1+A} \frac{\overline{F}(x)}{x} \quad \text{and} \quad \overline{F}(x) \leq \frac{C}{x^A}, \quad x > x_0, \quad \text{for some constant } C > 0.$$

**Proof.** Note that for  $x > x_0$ ,

$$\int_x^\infty \frac{f(u)}{u} du = - \int_x^\infty \frac{1}{u} d\overline{F}(u) = \frac{\overline{F}(x)}{x} - \int_x^\infty \frac{\overline{F}(u)}{u^2} du \geq \frac{\overline{F}(x)}{x} - \frac{1}{A} \int_x^\infty \frac{f(u)}{u} du.$$

The last inequality is due to (3). Hence

$$\left(1 + \frac{1}{A}\right) \int_x^\infty \frac{f(u)}{u} du \geq \frac{\bar{F}(x)}{x}.$$

On the other hand, for  $x > x_0$ ,

$$\begin{aligned} \log \bar{F}(x) &= - \int_0^x f(t)/\bar{F}(t) dt = - \int_0^{x_0} f(t)/\bar{F}(t) dt - \int_{x_0}^x f(t)/\bar{F}(t) dt \\ &\equiv C_0 - \int_{x_0}^x f(t)/\bar{F}(t) dt \leq C_0 - \int_{x_0}^x A/t dt = C_0 + A \log x_0 - A \log x. \end{aligned}$$

Therefore,  $\bar{F}(x) \leq C/x^A$ ,  $x > x_0$ , where  $C = x_0^A e^{C_0}$ . Lemma 4 is proved.

**Remark 5.** After deriving in Lemma 4 a lower bound for  $\int_x^\infty (f(u)/u) du$  we have the following upper bound for arbitrary density  $f$  on  $\mathbb{R}^+$ :

$$\int_x^\infty \frac{f(u)}{u} du \leq \frac{1}{x} \int_x^\infty f(u) du = \frac{\bar{F}(x)}{x}, \quad x > 0.$$

**Proof of Theorem 5.** The density  $g_n$  of  $Y_n$  is expressed as follows:

$$g_n(x) = \int_0^\infty \int_0^\infty \dots \int_0^\infty \frac{f(u_1)}{u_1} \frac{f(u_2)}{u_2} \dots \frac{f(u_{n-1})}{u_{n-1}} f\left(\frac{x}{u_1 u_2 \dots u_{n-1}}\right) du_1 du_2 \dots du_{n-1}$$

for  $x > 0$ . Hence  $g_n(x) > 0$  and decreases in  $x \in (0, \infty)$ . For any  $a > 0$ , we have

$$\begin{aligned} g_n(x) &\geq \int_a^\infty \int_a^\infty \dots \int_a^\infty \frac{f(u_1)}{u_1} \frac{f(u_2)}{u_2} \dots \frac{f(u_{n-1})}{u_{n-1}} f\left(\frac{x}{u_1 u_2 \dots u_{n-1}}\right) du_1 du_2 \dots du_{n-1} \\ &\geq \int_a^\infty \int_a^\infty \dots \int_a^\infty \frac{f(u_1)}{u_1} \frac{f(u_2)}{u_2} \dots \frac{f(u_{n-1})}{u_{n-1}} f\left(\frac{x}{a^{n-1}}\right) du_1 du_2 \dots du_{n-1} \\ &= f\left(\frac{x}{a^{n-1}}\right) \left(\int_a^\infty \frac{f(u)}{u} du\right)^{n-1}, \quad x > 0. \end{aligned} \tag{5}$$

The above second inequality follows from the monotone property of  $f$ . Taking  $a = x^{1/n} > x_0$ , we have, by (3)–(5) and Lemma 4, that

$$\begin{aligned} g_n(x) &\geq f(x^{1/n}) \left(\int_{x^{1/n}}^\infty \frac{f(u)}{u} du\right)^{n-1} \geq f(x^{1/n}) \left(\frac{A}{1+A} \frac{\bar{F}(x^{1/n})}{x^{1/n}}\right)^{n-1} \\ &\geq \left(\frac{A}{1+A}\right)^{n-1} x^{-(1-1/n)} \frac{f(x^{1/n})}{\bar{F}(x^{1/n})} (\bar{F}(x^{1/n}))^n \\ &\geq C_n x^{\gamma/n-1} e^{-\alpha x^{\beta/n}}, \end{aligned}$$

where  $C_n = \left(\frac{A}{1+A}\right)^{n-1} A B^n$ . Therefore, the Krein quantity for  $g_n$  is as follows:

$$\begin{aligned} \mathbf{K}[g_n] &= \int_0^\infty \frac{-\log g_n(x^2)}{1+x^2} dx = \int_0^{x_0^n} \frac{-\log g_n(x^2)}{1+x^2} dx + \int_{x_0^n}^\infty \frac{-\log g_n(x^2)}{1+x^2} dx \\ &\leq (-\log g_n(x_0^{2n})) \int_0^{x_0^n} \frac{1}{1+x^2} dx + \int_{x_0^n}^\infty \frac{-\log g_n(x^2)}{1+x^2} dx < \infty \quad \text{if } n > 2\beta. \end{aligned}$$

This in turn implies that  $Y_n$  is M-indet for  $n > 2\beta$  (see, e.g., Lin [12], Theorem 3). The proof is complete.

### 10. Part (d) of Basic Example

Let  $\xi \sim GG(\alpha, \beta, 1)$  with the density  $f(x) = ce^{-\alpha x^\beta}$ ,  $x \geq 0$ , where  $\alpha, \beta > 0$  and  $c$  is a norming constant. We claim that for  $n > 2\beta$ , the product  $Y_n = \xi_1 \xi_2 \cdots \xi_n$  is M-indet. To see this, note that  $f(x)/\bar{F}(x) \approx \alpha\beta x^{\beta-1}$  and  $\bar{F}(x) \approx [c/(\alpha\beta)]x^{1-\beta}e^{-\alpha x^\beta}$  as  $x \rightarrow \infty$ . Then the density function  $f$  satisfies the conditions (i) and (ii) in Theorem 5 and hence  $Y_n$  is M-indet if  $n > 2\beta$ .

For example, if  $\xi$  has the exponential distribution with density  $f(x) = e^{-x}$ ,  $x \geq 0$ , then the product  $Y_n = \xi_1 \xi_2 \cdots \xi_n$  is M-indet for  $n \geq 3$  as mentioned before.

If  $\xi$  has the half-normal distribution with density  $f(x) = \sqrt{2/\pi}e^{-x^2/2}$ ,  $x \geq 0$ , then  $Y_n = \xi_1 \xi_2 \cdots \xi_n$  is M-indet for  $n \geq 5$ . By words: The product of two, three or four half-normal random variables is M-det, while the product of five or more such variables is M-indet.

In summary, we have the following result about  $GG(\alpha, \beta, \gamma)$  with  $\gamma = 1$ .

**Lemma 5.** Let  $n \geq 2$ ,  $X_n = \xi^n$  and  $Y_n = \xi_1 \cdots \xi_n$ , where  $\xi, \xi_1, \dots, \xi_n$  are independent random variables with the same distribution  $GG(\alpha, \beta, 1)$ , where  $\alpha, \beta > 0$ . Then the power  $X_n$  is M-det iff the product  $Y_n$  is M-det and this is true iff  $n \leq 2\beta$ .

We now consider the general case  $\gamma > 0$ .

**Theorem 6.** Let  $n \geq 2$ ,  $X_n = \xi^n$  and  $Y_n = \xi_1 \cdots \xi_n$ , where  $\xi, \xi_1, \dots, \xi_n$  are independent random variables with the same distribution  $GG(\alpha, \beta, \gamma)$ , and  $\alpha, \beta, \gamma > 0$ . Then  $X_n$  is M-det iff  $Y_n$  is M-det and this is true iff  $n \leq 2\beta$ . In other words, both  $X_n$  and  $Y_n$  have the same moment determinacy property.

**Proof.** Define  $\eta = \xi^\gamma$ ,  $\eta_i = \xi_i^\gamma$ ,  $i = 1, 2, \dots, n$ ,  $X_n^* = \eta^n = (\xi^n)^\gamma = X_n^\gamma$  and  $Y_n^* = \eta_1 \eta_2 \cdots \eta_n = (\xi_1 \xi_2 \cdots \xi_n)^\gamma = Y_n^\gamma$ . Since  $\eta \sim GG(\alpha, \beta/\gamma, 1)$ , we have, by Lemma 5,  $X_n^*$  is M-det iff  $Y_n^*$  is M-det iff  $n \leq 2\beta/\gamma$ . Next, note that for each  $x > 0$ , we have  $\mathbf{P}[X_n^* > x] = \mathbf{P}[X_n > x^{1/\gamma}]$  and  $\mathbf{P}[Y_n^* > x] = \mathbf{P}[Y_n > x^{1/\gamma}]$ . This implies that any distributional property shared by  $X_n^*$  and  $Y_n^*$  can be transferred to a similar property shared by  $X_n$  and  $Y_n$ , and vice versa. Therefore,  $X_n$  is M-det iff  $Y_n$  is M-det iff  $n \leq 2\beta$ , because  $X_n$  is M-det iff  $n \leq 2\beta$  (see, e.g., Pakes and Khattree [21]). The proof is complete.

### 11. Second example

Some of the above results or illustrations involve the generalized gamma distribution  $GG$ . It is useful to have a moment determinacy characterization for non- $GG$  distributions. Here is an example based on the half-logistic distribution, clearly, not a  $GG$  one.

**Statement.** Suppose  $\xi$  is a random variable following the half-logistic distribution, i.e., its density is

$$f(x) = \frac{2e^{-x}}{(1 + e^{-x})^2}, \quad x \geq 0.$$

And let the power  $X_n$  and the product  $Y_n$  be defined as above. Then  $X_n$  is M-det iff  $Y_n$  is M-det and this is true iff  $n \leq 2$ . This means that for each  $n$ , the two random variables  $X_n$  and  $Y_n$  share the same moment determinacy property.

**Proof.** (i) The claim that  $X_n$  is M-det iff  $n \leq 2$  follows from results in Lin and Huang [14] who actually prove that for any real  $s > 0$ , the power  $\xi^s$  is M-det iff  $s \leq 2$ . Let us give here an alternative proof. The density  $h_s$  of  $\xi^s$  is

$$h_s(z) = \frac{2}{s} z^{1/s-1} \frac{e^{-z^{1/s}}}{(1 + e^{-z^{1/s}})^2}, \quad z \geq 0.$$

Using the inequality:  $1/4 \leq (1 + e^{-x})^{-2} \leq 1$  for  $x \geq 0$ , we find two-sided bounds for the moments of  $\xi^s$ :

$$\frac{1}{2} \Gamma(ks + 1) \leq \mathbf{E}[(\xi^s)^k] \leq \int_0^\infty \frac{2}{s} z^{k+1/s-1} e^{-z^{1/s}} dz = 2\Gamma(ks + 1).$$

Therefore the growth of moments is

$$\frac{\mathbf{E}[(\xi^s)^{k+1}]}{\mathbf{E}[(\xi^s)^k]} \leq 4 \cdot \frac{\Gamma((k+1)s + 1)}{\Gamma(ks + 1)} \approx 4s^s (k+1)^s \quad \text{as } k \rightarrow \infty.$$

By Theorem 1, this implies that  $\xi^s$  is M-det if  $s \leq 2$ . On the other hand, we have

$$\frac{\mathbf{E}[(\xi^s)^{k+1}]}{\mathbf{E}[(\xi^s)^k]} \geq \frac{1}{4} \cdot \frac{\Gamma((k+1)s + 1)}{\Gamma(ks + 1)} \approx \frac{1}{4} s^s (k+1)^s \quad \text{as } k \rightarrow \infty.$$

The moment condition in Theorem 4 is satisfied if  $s > 2$ . It remains now to check the validity of condition (2) for the density  $h_s$ . We have

$$\begin{aligned} L_{h_s}(z) &:= -\frac{zh'_s(z)}{h_s(z)} \\ &= 1 - \frac{1}{s} + \frac{1}{s} z^{1/s} - \frac{2}{s} z^{1/s} \frac{e^{-z^{1/s}}}{1 + e^{-z^{1/s}}} \nearrow \infty \text{ ultimately as } z \rightarrow \infty. \end{aligned}$$

Hence, if  $s > 2$ ,  $\xi^s$  is M-indet.

(ii) It remains to prove that  $Y_n$  is M-det iff  $n \leq 2$ .

(Sufficiency) As in part (i), we have

$$\frac{1}{2} \Gamma(k + 1) \leq \mathbf{E}[\xi^k] = 2\Gamma(k + 1).$$

Therefore,  $\mathbf{E}[\xi^{k+1}]/\mathbf{E}[\xi^k] = O(k + 1)$  as  $k \rightarrow \infty$ . By Theorem 2, we conclude that  $Y_n$  is M-det if  $n \leq 2$ .

(Necessity) Note that  $\bar{F}(x) = \mathbf{P}[\xi > x] = 2e^{-x}/(1 + e^{-x}) \geq e^{-x}$ ,  $x \geq 0$ , and  $f(x)/\bar{F}(x) = 1/(1 + e^{-x}) \geq 1/2$ ,  $x \geq 0$ . Therefore, taking  $\beta = 1$  in Theorem 5, we conclude that  $Y_n$  is M-indet if  $n > 2$ . By words: The product of three or more half-logistic random variables is M-indet. The proof is complete.

## 12. Determinacy of the product of random number of random variables

Next, we consider a random-number product of  $\xi_i$ ,  $i = 1, 2, \dots$ , independent copies of  $\xi \geq 0$ . Denote  $X_N = \xi^N$  and  $Y_N = \xi_1 \xi_2 \cdots \xi_N$ , where  $N$ , independent of  $\{\xi_i\}_{i=1}^\infty$ , is a positive integer-valued random variable with  $\mathbf{P}[N = n] = p_n$ ,  $n = 1, 2, \dots$ . Then we note first that the moment inequality (1) valid for any fixed  $n$  can be extended as follows. For all  $s > 0$ , we have, by Lyapunov's inequality and the total expectation formula, that

$$\begin{aligned} \mathbf{E}[X_N^s] &= \sum_{n=1}^{\infty} p_n \mathbf{E}[X_n^s] = \sum_{n=1}^{\infty} p_n \mathbf{E}[(\xi^n)^s] = \sum_{n=1}^{\infty} p_n \mathbf{E}[\xi^{ns}] \\ &\geq \sum_{n=1}^{\infty} p_n (\mathbf{E}[\xi^s])^n = \sum_{n=1}^{\infty} p_n \mathbf{E}[Y_n^s] = \mathbf{E}[Y_N^s]. \end{aligned} \quad (6)$$

By using inequality (6), Proposition 1 can be extended as follows.

**Proposition 2.** (i) If the random power  $X_N = \xi^N$  satisfies Carleman's condition, then so does the random product  $Y_N = \xi_1 \cdots \xi_N$ .  
(ii) If  $X_N$  satisfies Cramér's condition, then so does  $Y_N$ .  
(iii) If  $X_N$  satisfies Hardy's condition, then so does  $Y_N$ .

Moreover, Theorem 5 can also be extended to random products as follows.

**Theorem 7.** Let  $N$  be a positive integer-valued random variable with  $\mathbf{P}[N = n] = p_n$ ,  $n = 1, 2, \dots$ , and let  $N$  be independent of  $\xi_i$ ,  $i = 1, 2, \dots$ . Then, in addition to the assumptions of Theorem 5, if there exists an index  $n^*$  such that  $n^* > 2\beta$  and  $p_{n^*} > 0$ , then the random product  $Y_N$  has a finite Krein quantity and is M-indet.

**Proof.** Let  $\tilde{g}$  and  $g_n$  be the density functions of  $Y_N$  and  $Y_n$ , respectively. Then

$$\tilde{g}(x) = \sum_{n=1}^{\infty} p_n g_n(x), \quad x > 0.$$

For  $n^* > 2\beta$  with  $p_{n^*} > 0$ ,  $\tilde{g}(x) \geq p_{n^*} g_{n^*}(x)$ ,  $x > 0$ , and the Krein quantity for  $g_{n^*}$  is finite by Theorem 4. Therefore, the Krein quantity for  $\tilde{g}$  is

$$\begin{aligned} \mathbf{K}[\tilde{g}] &= \int_0^\infty \frac{-\log \tilde{g}(x^2)}{1+x^2} dx \leq \int_0^\infty \frac{-\log (p_{n^*} g_{n^*}(x^2))}{1+x^2} dx \\ &\leq \int_0^\infty \frac{-\log p_{n^*}}{1+x^2} dx + \int_0^\infty \frac{-\log g_{n^*}(x^2)}{1+x^2} dx \\ &= \int_0^\infty \frac{-\log p_{n^*}}{1+x^2} dx + \mathbf{K}[g_{n^*}] < \infty. \end{aligned}$$

Hence  $Y_N$  is M-indet. The proof is complete.

The main result in Ostrovska and Stoyanov [19], see also Berg [3], is proved for products of a fixed number of exponentials. This can now be extended to random products.



**Corollary 3.** Let  $\xi_1, \xi_2, \dots$ , be a sequence of independent standard exponential random variables, and let  $N$  be a positive integer-valued random variable independent of  $\{\xi_i\}_{i=1}^\infty$ . If  $\mathbf{P}[N = n^*] > 0$  for some  $n^* \geq 3$ , then the random product  $Y_N$  has a finite Krein quantity and is M-indet.

### 13. Maximum order statistics with random sample size

Instead of the random power  $X_N$  and the random product  $Y_N$ , we now compare  $X = \xi$  and  $Z_N = \max\{\xi_1, \xi_2, \dots, \xi_N\}$ , where  $N, \xi, \xi_i, i = 1, 2, \dots$ , are defined as in Section 12. Let  $\psi$  be the probability generating function of  $N$ , namely,

$$\psi(t) = \sum_{n=1}^{\infty} p_n t^n, \quad t \in [0, 1].$$

Then we have the following.

**Lemma 6.**  $p_1 \leq \psi'(t) \leq \mathbf{E}[N]$ ,  $t \in [0, 1]$ .

**Proof.** Since the derivative of the function  $\psi$ ,

$$\psi'(t) = \sum_{n=1}^{\infty} n p_n t^{n-1} = p_1 + \sum_{n=2}^{\infty} n p_n t^{n-1}, \quad t \in [0, 1],$$

is increasing on  $[0, 1]$ , we have

$$p_1 = \psi'(0) \leq \psi'(t) \leq \psi'(1) = \sum_{n=1}^{\infty} n p_n = \mathbf{E}[N] \quad \text{for } t \in [0, 1].$$

This completes the proof.

**Lemma 7.** For each real  $s > 0$ ,  $p_1 \mathbf{E}[X^s] \leq \mathbf{E}[Z_N^s] \leq \mathbf{E}[N] \mathbf{E}[X^s]$ .

**Proof.** Let  $F$  and  $G$  be the distributions of  $X$  and  $Z_N$ , respectively. Then

$$G(x) = \sum_{n=1}^{\infty} p_n F^n(x) = \psi(F(x)), \quad x \geq 0.$$

Define the quantile function of  $F$  by  $F^{-1}(t) = \inf\{x : F(x) \geq t\}$ ,  $t \in (0, 1)$ . Write

$$\mathbf{E}[Z_N^s] = \int_0^\infty x^s dG(x) = \int_0^\infty x^s \psi'(F(x)) dF(x) = \int_0^1 (F^{-1}(t))^s \psi'(t) dt.$$

Since  $\mathbf{E}[X^s] = \int_0^1 (F^{-1}(t))^s dt$ , we refer to Lemma 6 and this completes the proof of Lemma 7.

Let us use Lemma 7 and summarize our findings regarding the moment determinacy of the random variable  $X = \xi$  and the maximum  $Z_N = \max\{\xi_1, \dots, \xi_N\}$ .

**Proposition 3.** In addition to the above setting, assume  $p_1 > 0$  and  $\mathbf{E}[N] < \infty$ . Then  $X$  and  $Z_N$  have the same number of finite moments. Moreover, if all moments of  $X$  are finite, the following statements hold:

- (i) If  $X$  satisfies Carleman's condition, so does  $Z_N$ , and vice versa.
- (ii) If  $X$  satisfies Cramér's condition, so does  $Z_N$ , and vice versa.
- (iii) If  $X$  satisfies Hardy's condition, so does  $Z_N$ , and vice versa.

#### 14. Equilibrium transform

It is interesting to consider another transform of  $X$  or  $F$ , instead of powers and products. Here is one possibility. Let  $0 \leq X \sim F$  with finite moments of all orders and  $\mathbf{E}[X] > 0$ . Define the equilibrium transform of  $F$  by

$$F_{(1)}(x) = \frac{1}{\mathbf{E}[X]} \int_0^x \bar{F}(t) dt, \quad x \geq 0$$

(see Cox [5], p. 64). Then  $F_{(1)}$  is a distribution function and let  $X_{(1)}$  be a random variable,  $X_{(1)} \sim F_{(1)}$ . Then we have

$$\mathbf{E}[X_{(1)}^s] = \frac{1}{s+1} \frac{\mathbf{E}[X^{s+1}]}{\mathbf{E}[X]} \quad \text{for any real } s > -1 \quad (7)$$

(see Lin [13], Lemma 3). Note that  $F_{(1)}$  is absolutely continuous with density  $f_{(1)}(x) = \bar{F}(x)/\mathbf{E}[X]$ ,  $x \geq 0$ , regardless of continuity of  $F$ . Important is to tell that this transformation, from  $F$  to  $F_{(1)}$ , is not one-to-one; see, e.g., Huang and Lin [11].

From (7) we see that if  $X_{(1)}$  is M-det then so is  $X$ , but the converse is not true in general. Namely, it is possible that  $X$  is M-det while  $X_{(1)}$  is M-indet (see, e.g., Berg and Thill [4] and Berg [2]). Therefore,  $X$  and  $X_{(1)}$  in general do not share the same moment determinacy property. By the moment relationship (7), we establish the next result, Proposition 4, in which part (iv) implies that both  $X$  and  $X_{(1)}$  have the same growth rate of the moments.

**Proposition 4.** Let  $X$  and  $X_{(1)}$  be defined as above.

- (i) If  $X_{(1)}$  satisfies Carleman's condition, then so does  $X$ .
- (ii) If  $X_{(1)}$  satisfies Cramér's condition, then so does  $X$ , and vice versa.
- (iii) If  $X_{(1)}$  satisfies Hardy's condition, then so does  $X$ .
- (iv)  $\mathbf{E}[X_{(1)}^{s+1}]/\mathbf{E}[X_{(1)}^s] = \frac{s+1}{s+2} \mathbf{E}[X^{s+2}]/\mathbf{E}[X^{s+1}]$  for any real  $s > -1$ .

**Proof.** (i) By (7) and Lemma 2(ii), we have  $\mathbf{E}[X_{(1)}^n] \geq \frac{1}{n+1} \mathbf{E}[X^n]$  and hence

$$\sum_{n=1}^{\infty} (\mathbf{E}[X_{(1)}^n])^{-1/(2n)} \leq \sum_{n=1}^{\infty} (n+1)^{1/(2n)} (\mathbf{E}[X^n])^{-1/(2n)}.$$

Note that  $\lim_{n \rightarrow \infty} (n+1)^{1/(2n)} = 1$ . Therefore, if  $X_{(1)}$  satisfies Carleman's condition, then so does  $X$ .

(ii) For  $c > 0$ , we have

$$\begin{aligned} \mathbf{E}[\exp(cX_{(1)})] &= \sum_{n=0}^{\infty} \frac{c^n}{n!} \mathbf{E}[X_{(1)}^n] = \frac{1}{c\mathbf{E}[X]} \sum_{n=0}^{\infty} \frac{c^{n+1}}{(n+1)!} \mathbf{E}[X^{n+1}] \\ &= \frac{1}{c\mathbf{E}[X]} \sum_{n=1}^{\infty} \frac{c^n}{n!} \mathbf{E}[X^n] = \frac{\mathbf{E}[\exp(cX)] - 1}{c\mathbf{E}[X]}. \end{aligned}$$

Therefore, if  $X_{(1)}$  satisfies Cramér's condition, then so does  $X$ , and vice versa.  
(iii) For  $c > 0$ , we have

$$\begin{aligned}\mathbf{E} [\exp(c\sqrt{X_{(1)}})] &= \sum_{n=0}^{\infty} \frac{c^n}{n!} \mathbf{E}[X_{(1)}^{n/2}] = \frac{2}{c^2 \mathbf{E}[X]} \sum_{n=0}^{\infty} (n+1) \frac{c^{n+2}}{(n+2)!} \mathbf{E}[X^{(n+2)/2}] \\ &= \frac{2}{c^2 \mathbf{E}[X]} \sum_{n=2}^{\infty} (n-1) \frac{c^n}{n!} \mathbf{E}[X^{n/2}] \\ &\geq \frac{2}{c^2 \mathbf{E}[X]} \left( \mathbf{E}[\exp(c\sqrt{X})] - 1 - c\mathbf{E}[\sqrt{X}] \right).\end{aligned}$$

Therefore, if  $X_{(1)}$  satisfies Hardy's condition, then so does  $X$ .

(iv) The proof follows immediately from (7). The proof of Proposition 4 is complete.

We now give conditions for the moment determinacy and indeterminacy of  $X_{(1)}$  in terms of the moments  $m_k = \mathbf{E}[X^k]$  of  $X$ .

**Theorem 8.** (i) If  $m_{k+1}/m_k = O((k+1)^2)$  as  $k \rightarrow \infty$ , then  $X$  and  $X_{(1)}$  are M-det.  
(ii) Let  $m_{k+1}/m_k \geq C(k+1)^{2+\varepsilon}$  for all large  $k$ , where  $C$  and  $\varepsilon$  are positive constants. Assume further that  $X$  has a density  $f$  which satisfies the condition: for some  $x_0 > 0$ ,  $\overline{F}$  is positive on  $[x_0, \infty)$  and

$$L_{f_{(1)}}(x) := -\frac{xf'_{(1)}(x)}{f_{(1)}(x)} = \frac{xf(x)}{\overline{F}(x)} \nearrow \infty \text{ as } x_0 < x \rightarrow \infty.$$

Then  $X_{(1)}$  is M-indet.

**Proof.** The statements follow from Theorems 1 and 4, and Proposition 4(iv), because the following relation holds:

$$\mathbf{E}[X_{(1)}^{k+1}]/\mathbf{E}[X_{(1)}^k] \approx \mathbf{E}[X^{k+2}]/\mathbf{E}[X^{k+1}] \text{ as } k \rightarrow \infty.$$

The proof is complete.

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